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Analytical Derivation of Spectral Scalability in Self-Similar Multilayer Structures

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ABSTRACT

Optical spectra for fractal multilayer structures have been shown to possess scalability. The scaling relations, as well as analytical derivation of scalability on the basis of the structure's geometrical self-similarity, have been established.
Keywords: scaling laws, self-similarity, fractals, multilayers, Cantor structures, eigenvalue problem.

1. INTRODUCTION

The problem of electromagnetic wave propagation in multilayer media, or alternatively, the problem of quantum particle behavior in a stepwise energy profile is known to reduce to the eigenvalue problem for the Helmholtz (or Schrodinger) equation with a complex stepwise potential. However, analytical treatment of such a general case has not yet been developed and scientists have to resort to numerical methods, which, though capable of finding valid solutions, fail to provide much theoretical insight into the problem.

One of the ways that appear to be fruitful in gaining such understanding is to seek out and analyze particular cases of correlation between the geometrical properties of the structures and the properties of their eigenvalue spectrum.

The two extreme (and hence most studied) cases of multilayer structures are good examples. First, the spectra of periodic multilayers have been found to possess forbidden gaps, which are demonstrated to directly result from geometrical periodicity. Second, disordered dielectric media have been discovered to slow down, localize, and confine light waves traveling through them.

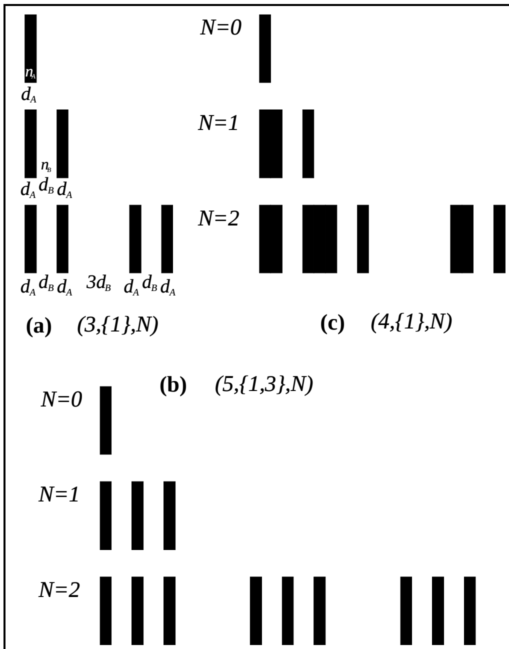

Figure 1. Sample fractal multilayers: (a) (3, {1}, N); (b) (5, {1, 3}, N); (c) (4, {1}, N).

Recent studies revealed that "intermediate" cases (non-periodic yet deterministic structures) may not only display properties of both extreme cases but also exhibit their own, special effects. For instance, it was found [1] that *quasiperiodic* (e.g., Fibonacci) multilayers have *self-similar* spectra, which represent Cantor sets, a well-known example of one-dimensional (1D) fractals. Thus it follows that spectral self-similarity is a characteristic property of geometrical quasiperiodicity.

In this paper, we have investigated another class of non-periodic but deterministic structures, the *fractal multilayers*. These self-similar structures appeared to have their optical spectra obeying *scaling laws* first noted by us in numerical computation [2], and the fact is a direct result of the geometrical self-similarity.

2. FRACTAL MULTILAYERS

As was mentioned above, *fractal* multilayers are those having features of both periodic and random ones. Although it is intuitively understood what to call a fractal multilayer, the strict definition thereof is not so clear. One way is to define *fractal multilayer structures* as those constructed according to a known fractal generation algorithm. However, this algorithm has to be stopped at some point in order to get a finite structure. Therefore, any structure obtained in this way is not a genuine fractal, but rather a *one-dimensional prefractal*, which can be regarded as another definition of a fractal multilayer.

A common example is a well-known Cantor stack generated using the "middle third removal" procedure [3] (see Fig.1a). However, this procedure can be generalized. The most straightforward way to do so is to complicate the removal routine, applying it not only to the middle third, but to arbitrary (yet similar from generation to generation) regions of the structure. Some variations are described in [3], and investigated in [2,4].

In this paper, we introduce even more general procedure. The algorithm starts with a ***seed*** assigned N=0. The seed is a single dielectric layer (label it A) with the index of refraction n_A and thickness d_A. The seed is then stacked together G times, and the layers are numbered in base G (starting with zero). Then, those parts whose numbers belong to a given subset of digits $\mathbf{C} \subset \{0,1,\ldots,G-1\}$ are removed and replaced with another material (labeled B), with the index of refraction n_B and thickness d_B. This ***replication-replacement*** (RR) procedure is then repeated for the resulting structure (which now consists of G layers and is assigned N=1), with the only difference that a group of G B-type layers is now used to replace the appropriate fragments. Repeating this RR procedure multiple times yields the desired fractal multilayer.

Here, an arbitrary integer $G>2$ together with the subset $\mathbf{C}$ form the ***generator*** of the structure, while the assigned number N (actually, the number of RR procedures applied) is the ***number of generations***. The whole structure can be referred to as a (G, $\mathbf{C}$, N)-structure. The total number of layers of such a structure is G^N, $(G\text{-}C)^N$ of which are A-layers, C being the number of members in $\mathbf{C}$. For convenience of further considerations, several adjacent layers of the same material (A or B) are considered separate layers.

One can easily see this algorithm can produce (i) "usual" middle-third (3, {1}, N), pentadic (5, {1, 3}, N) and higher-G Cantor structures (G=2n-1, $\mathbf{C}$={1, 3 … G-2}, N) (see Fig. 1a, b); (ii) non-symmetric structures described in [3], e.g., (5, {1, 2}, N), (6, {1, 3, 4}, N) and like that; (iii) generalized Cantor bars spoken of in [4] of the type (G, {M, M+1,…,G-M-1}, N) (see Fig. 1c).

The total thickness of the (G, $\mathbf{C}$, N) structure depends on N as

$$\Delta_N = (G-C)^N d_A + \left(G^N - (G-C)^N\right) d_B \tag{1}$$

and can be written as

$$\Delta_N \equiv (G-C)\Delta_{N-1} + C\tilde{\Delta}_{N-1}\,. \tag{2}$$

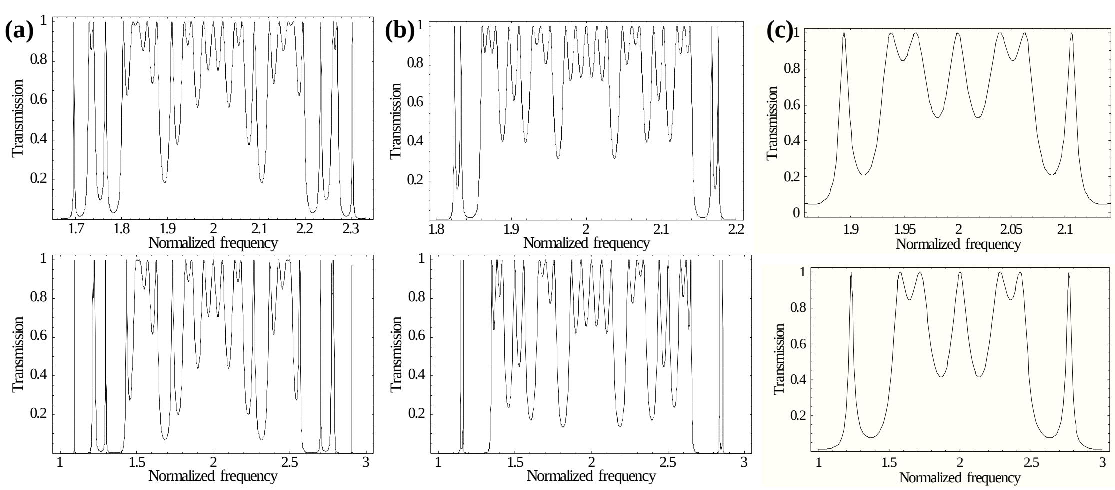

Figure 2. Scalability of optical spectra for fractal multilayers. Central part of the spectrum for(3,{1},4) scaled by S=3 (a, top) versus full spectrum for (3,{1},4) (a ,bottom); the same for (5,{1,3},3), S=5, and (5,{1,3},2) (b); the same for (7,{1,5},2), S=7, and (7,{1,5},1) (c). Carefully comparing the plots in the columns, one can see that scalability is apparent yet not exact.

3. NUMERICAL OBSERVATION OF SPECTRAL SCALABILITY

For all numerical calculations, the constituent layers were chosen to satisfy the *quarter wave condition*, i.e.,

$$n_A d_A = n_B d_B = l_0/4\,. \tag{3}$$

This condition dictates that the spectra are periodic with respect to frequency, the period equal to $2\omega_0=4\pi c/\lambda_0$. This is very convenient, since it provides a natural way to normalize the frequency scale as well as allows only *one* period of spectrum to be referred to as "spectrum", which is what will be done hereafter.

We have found earlier [2] that if one magnifies the central region (i.e., located around the central frequency) of the spectrum of a (*G*, **C**, *N*)-structure by a factor of *G* along the frequency axis, this central region will match the spectrum of a (*G*, **C**, *N*-1) almost perfectly. This *spectral scalability* was first noted for Cantor structures [2] (see Fig. 2a,b), but subsequent research has revealed that other fractal multilayers also exhibit spectral scalability (see Fig. 2c). What is more, the *scaling factor S* [i.e., the factor by which one has to magnify the central part of the (*G*, **C**, N_1) stack spectrum for matching that of (*G*, **C**, $N_2< N_1$) stack] is the same in this case and again equals

$$S = G^{N_1-N_2}\,. \tag{4}$$

It was already reported [2] that the origin of the spectral scalability is geometrical self-similarity of fractal multilayers. However, the only consideration to back up this claim so far has been the fact that the scaling factor in (4) exactly equals the geometrical factor of self-similarity, which is clearly seen from the construction procedure. In what follows, we provide a more direct and convincing analytical proof. The full calculations will be made using the simplest case (3, {1}, *N*) as an example, with the outline for generalization to the whole class of fractal multilayers.

4. ANALYTICAL DERIVATION OF SCALABILITY

4.1 The Sun-Jaggard method of computation and its modification for arbitrary layer thicknesses

To analytically calculate the spectra of Cantor multilayers, we made use of so-called *self-similarity method of calculation*, proposed by Sun and Jaggard in [4] and based on the structure being self-similar. However, the method was initially designed only for the case $d_A= d_B$, and thus had to be modified to comply with the condition (3). According to this modified method, the reflection and transmission coefficients for the structures of N^{th} and $N\text{-}1^{\text{th}}$ generation are related as

$$R_{N+1}(\Delta_{N+1},\omega)=g_r\left[R_N(\Delta_N,\omega),T_N(\Delta_N,\omega),\tilde{\Delta}_N,\omega\right],\quad T_{N+1}(\Delta_{N+1},\omega)=g_t\left[R_N(\Delta_N,\omega),T_N(\Delta_N,\omega),\tilde{\Delta}_N,\omega\right], \tag{4}$$

where Δ_N is the overall thickness of the structure and $\tilde{\Delta}_N$ is as defined in equation (2). The functions

$$g_r(x,y,d,\omega)=x+\frac{xy^2\exp\left[\frac{2i}{c}\omega n_B d\right]}{1-x^2\exp\left[\frac{2i}{c}\omega n_B d\right]},\quad g_t(x,y,d,\omega)=\frac{y^2\exp\left[\frac{i}{c}\omega n_B d\right]}{1-x^2\exp\left[\frac{2i}{c}\omega n_B d\right]} \tag{5}$$

were obtained in [4] using the effective medium formalism. The boundary conditions for these recurrent relations are reflection and transmission spectra of a single layer, that is,

$$R_0(\Delta_0,\omega)=-r+\frac{rtt'\exp\left[\frac{2i}{c}\omega n_A\Delta_0\right]}{1-r^2\exp\left[\frac{2i}{c}\omega n_A\Delta_0\right]},\qquad T_0(\Delta_0,\omega)=\frac{tt'\exp\left[\frac{i}{c}\omega n_A\Delta_0\right]}{1-r^2\exp\left[\frac{2i}{c}\omega n_A\Delta_0\right]} \tag{6}$$

where $r=(n_A-n_B)/(n_A+n_B)$, $t=2n_B/(n_A+n_B)$, $t=2n_B/(n_A+n_B)$, $t'=2n_A/(n_A+n_B)$, and $\Delta_0=d_A$.

4.2 Scale transformation of analytically derived spectra

To proceed with the analysis of scalability, one needs to compare the following coefficients: $T_{N+1}(\Delta_{N+1}, \omega/3)$ and $T_N(\Delta_N, \omega)$. From equations (1) and (2) it follows that for G=3 and C=1 the thickness $\tilde{\Delta}_N = 3^N d_B$. Thus, expanding the quantities using equations (4) and (5), we have

$$T_{N+1}\left(\Delta_{N+1},\tfrac{\omega}{3}\right)=\frac{T_N^2\left(\Delta_N,\frac{\omega}{3}\right)\exp\left[\frac{i}{c}\frac{\omega}{3}n_B\cdot 3^N d_B\right]}{1-R_N^2\left(\Delta_N,\frac{\omega}{3}\right)\exp\left[\frac{2i}{c}\frac{\omega}{3}n_B\cdot 3^N d_B\right]},\quad T_N(\Delta_N,\omega)=\frac{T_{N-1}^2(\Delta_{N-1},\omega)\exp\left[\frac{i}{c}\omega n_B\cdot 3^{N-1}d_B\right]}{1-R_{N-1}^2(\Delta_{N-1},\omega)\exp\left[\frac{2i}{c}\omega n_B\cdot 3^{N-1}d_B\right]}, \tag{7}$$

and we see that the exponents in are exactly the same. The sole difference between the quantities in (7) lies in the coefficients, namely, $T_N(\Delta_N, \omega/3)$, $R_N(\Delta_N, \omega/3)$ and $T_{N-1}(\Delta_{N-1}, \omega)$, $R_{N-1}(\Delta_{N-1}, \omega)$, respectively. But these coefficients can in turn be expanded in just the same manner, and again, the exponents will be equal, the difference present in the coefficients. This way, the expansion can be traced repeatedly and all the frequency-dependent exponents that appear along the way are equal. However, the tracing has its limits, because once N=0, the coefficients can no longer be expanded.

4.3 Analysis of initial conditions

So, the equality in the quantities (4) reduces to the conditions

$$T_1\left(\Delta_1, \tfrac{\omega}{3}\right) = T_0\left(\Delta_0, \omega\right), \qquad R_1\left(\Delta_1, \tfrac{\omega}{3}\right) = R_0\left(\Delta_0, \omega\right) \tag{8}$$

which, on applying equations (4-6) and performing the substitution

$$\varepsilon \equiv \exp\left[\tfrac{i}{c}\tfrac{\omega}{3} n_A d_A\right], \qquad \varepsilon' \equiv \exp\left[\tfrac{i}{c}\tfrac{\omega}{3} n_B d_B\right] \tag{9}$$

yields that for the conditions (8) to hold, and hence, the scalability to be apparent, it is required that

$$\frac{tt'\varepsilon'^3}{1-r^2\varepsilon'^6} = \frac{\left(\dfrac{tt'\varepsilon'}{1-r^2\varepsilon'^2}\right)^2 \varepsilon}{1-\left(\dfrac{rtt'\varepsilon'^2}{1-r^2\varepsilon'^2}-r\right)^2 \varepsilon^2}, \qquad \frac{rtt'\varepsilon'^6}{1-r^2\varepsilon'^6} = \frac{rtt'\varepsilon'^2}{1-r^2\varepsilon'^2} + \frac{\left(\dfrac{rtt'\varepsilon'^2}{1-r^2\varepsilon'^2}-r\right)\left(\dfrac{tt'\varepsilon'}{1-r^2\varepsilon'^2}\right)^2 \varepsilon^2}{1-\left(\dfrac{rtt'\varepsilon'^2}{1-r^2\varepsilon'^2}-r\right)^2 \varepsilon^2}. \tag{10}$$

It can be seen that these conditions cannot hold for arbitrary ω. Nevertheless, we can take into account the refraction index contrast n_A/n_B rarely exceeds 2 in common multilayers and often is as much as 1.5. Hence, one can see from its definition that r is normally less than $0.2 \div 0.3$. This allows to use r as a small parameter and introduce the first approximation as

$$r^2 \approx 0 \Rightarrow tt' = 1 - r^2 \approx 1. \tag{11}$$

This immediately reduces both conditions (10) to a very simple form

$$\varepsilon' \approx \varepsilon, \tag{12}$$

which can be true for arbitrary ω if (and only if) $n_B d_B = n_A d_A$, none other than the quarter-wave condition (3).

5. CONCLUSIONS

We can see that the optical spectra for Cantor multilayers indeed appear to possess scalability, and indeed, it can be derived analytically using a method that directly employs geometrical self-similarity of a fractal structure. This lets us conclude that *indeed, spectral scalability is the result of geometrical self-similarity, and these properties accompany each other*. However, there are notes that have to be made.

The first one regards the generalization of the analytically obtained results to the whole class of fractal multilayers. This can be achieved by generalizing the Sun-Jaggard computation procedure, which is straightforward and thus is only a matter of time and mathematical effort.

Secondly, as seen from both analytical [see the approximation (12)] and numerical calculations (on careful inspection of Fig. 2), scalability is only ***approximate*** in real multilayers. However, we state this fact results from the structures under study being ***finite***. Had it been otherwise, i.e., if the number of generations approached infinity, the spectral scalability would be exact, as seen from the calculations, regardless of both the quarter wave condition (3) and the refraction index contrast approximation (11).

But what we have investigated are in fact prefractals rather than true fractals, and geometrical self-similarity in them is not exact as well. This disturbs the scalability effect in much the same way as it occurs in other types of media. For example, finite periodic structures cannot exhibit completely zero transmission in the bang gaps, and in finite disordered media light cannot be completely trapped. In this manner, N^{th} generation Cantor multilayers can be compared to N-period 1D photonic crystals, while it is commonly known that band gaps are usually apparent at a much larger N than was used for the plots in Fig. 2.

However, if certain conditions are satisfied, one can observe decent band gaps even in periodic multilayers with as many as four periods. An analogous statement seems to be true for scalability in fractal multilayers. But the conditions to be desired are directly opposite. As seen from approximation (11), scalability is best observed if the refraction index contract is smaller, while band structure is much more pronounced if the contrast is large enough. As regards the quarter wave condition (3), it seems to play an important part in the manifestation of characteristic spectral properties in both kinds of materials, making both phenomena more apparent to observe.

To summarize, both the condition (3) and the approximation (11) are not important theoretically, but are of great value practically, allowing to observe apparent scalability even with smaller N, as seen from Fig. 2.

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