Scaling properties of multilayer fractal structures

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1. Introduction

As regards light propagation in multilayer media in general, it should be noted that the majority of results here were obtained in the studies of the two opposite cases. On the one hand, the properties of perfectly periodic media (such as photonic crystals) have been generally described and understood, and those media are known to exhibit photonic band gaps and omnidirectional reflection, which offers a wide variety of applications [1-3]. On the other hand, there are numerous studies devoted to light localization and coherent backscattering in absolutely random multilayers (see the review [4] and references therein).

However, there are structures that behave much like 'disordered' but are constructed according to a deterministic procedure. They are called ***non-periodic deterministic (NPD) media***, and they constitute a separate field of research because they possess the properties of both periodic and random structures and also have some distinct features not found in traditional media. Not yet studied to a sufficient extent, NPD multilayer stacks lack a neat and generally accepted classification, but one can clearly distinguish several large groups in accordance with the algorithm used for the stack construction.

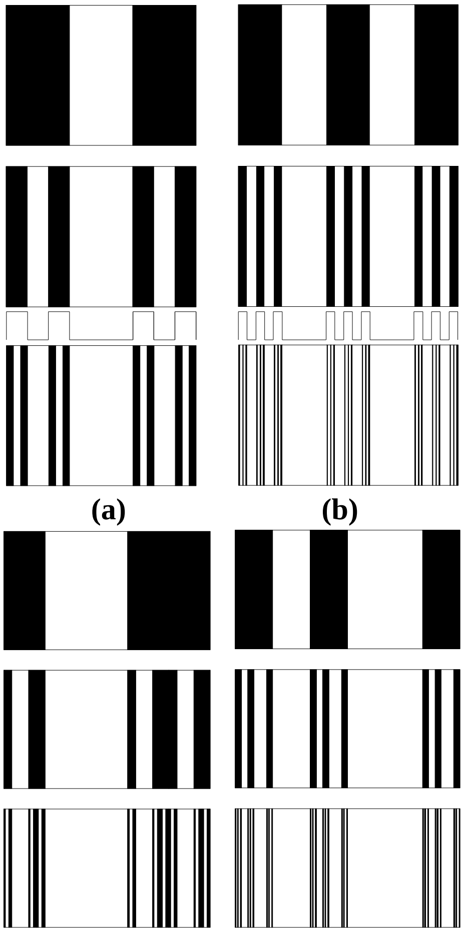
There are, for instance, ***substitutional lattices***, so called because they are 'generated' via a repeated substitution rule. These structures were first noticed and thus are best researched among NPD's. In terms of the present paper it is vital that substitutional lattices possessing quasi-periodicity (***Fibonacci multilayers***) have self-similar optical transmission spectra, and the frequencies of resonance peaks form a fractal Cantor set [5]. At the same time, there are NPD multilayers that are fractal by themselves. They are called *multilayer fractal structures* because they are constructed in a similar way to well-known fractal sets. These structures are much less studied (see for example [6-9]), and are subject to investigation in this paper, which is organized as follows. Section 2 deals with the definition of fractal multilayers and certain mathematical conventions used later on. Section 3 describes *spectral scalability*, one of the two effects found that indicate a strong connection between geometrical and spectral features of multilayers. Discussion and analysis of the other such effect, *sequential splitting*, is what follows in Section 4. Finally, Section 5 summarizes the paper and outlines ways for future investigation.

2. Multilayer fractal structures

As was mentioned above, *fractal* multilayers are those opposed to both periodic and random ones. Although it is intuitively understood what to call a fractal multilayer (see Fig. 1 for some examples), the strict definition thereof is not so clear. One way is to define *multilayer fractal structures* as those constructed according to a known fractal generation algorithm [9]. However, as will be shown below, this algorithm has to be stopped at some point in order to get a finite structure. Therefore, any structure obtained in this way is not a genuine fractal, but rather a ***one-dimensional prefractal***, which can be regarded as another definition of a multilayer fractal structure.

In this paper, we are primarily interested in *generalized Cantor multilayers* constructed as follows. The algorithm starts with a *seed* — a solid bulk of a dielectric material (label it A). The seed is then divided into G equal parts numbered in base G (starting with zero), and parts whose number belongs to a given subset of digits $\mathbf{C}$ are removed from the bulk and replaced with another material (labeled B), different from A in its index of refraction n. This division and replacement (DR) procedure is then executed multiple times over all the remaining inclusions of A as if they were individual seeds. Here, an arbitrary integer $G>2$ together with the subset $\mathbf{C}$ form the *generator* of the structure, while the number of DR procedures N is the *number of generations*. The whole structure will be referred to as a (G, $\mathbf{C}$, N)-structure. The total number of layers of such a structure is G^N, $(G\text{-}C)^N$ of which are A-layers, C being the number of members in $\mathbf{C}$. For convenience of further considerations, several adjacent layers of the same material (A or B) are considered separate layers.

One can easily see this algorithm can produce (i) 'usual' triadic (3, {1}, N), pentadic (5, {1, 3}, N) and higher-G Cantor structures (G=2n-1, $\mathbf{C}$={1, 3 … G-1}, N) (see Fig. 1a, b); (ii) non-symmetric



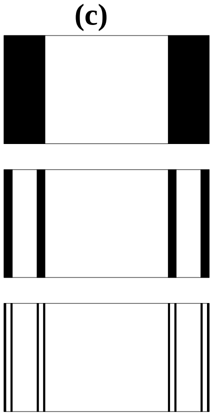


FIG. 1. Multilayer fractal structures of various types: Cantor multilayers (3,{1},*N*) **(a)** and (5,{1,3},*N*) **(b)**; non-symmetric stacks (5,{1,2},*N*) **(c)** and (6,{1,3,4},*N*) **(d)**; generalized Cantor bar (5,{1,2,3},*N*) **(e)**. All structures are shown for *N*=1,2,3 (*top to bottom*).

structures described in [10], e.g., (5, {1, 2}, *N*), (6, {1, 3, 4}, *N*) and like that, see Fig. 1c, d; (iii) generalized Cantor bars spoken of in [11] of the type (*G*, {*M*, *M*+1,…,*G*-*M*-1}, *N*) (see Fig. 1e).

It should be noted that the algorithm just described can be used in two slightly different versions. In the first one, the thickness of the ***whole*** structure remains the same for any *N* and equals the thickness of the initial seed, while the structure becomes 'finer' as *N* increases. It is this version that is shown in Fig. 1. In the other one, instead of the division of the seed into *G* parts, the whole structure is ***replicated*** *G* times, and those instances that nave numbers within **C** are replaced with a layer of B whose thickness varies with *N* according to the power law G^N. Fig. 1d explains this with more clarity. In this version it is the thickness of an ***individual*** layer that remains the same for any *N*, while the total thickness ***d*** depends on *N* as

$$d=(G-C)^N d_A+\left(G^N-(G-C)^N\right)d_B$$

where d_A and d_B are initial thicknesses of individual layers of A and B, respectively. (One can easily see (1) reduce to $d = G^N d_A$ when $d_A = d_B$). Now it is clear why it is necessary to keep *N* finite in both versions. In the first one, the structure remains finite but cannot be treated as a multilayer if the generations approaches infinity. In the second one, though the structure remains a multilayer, it is rendered infinite non-periodic non-random, and the techniques suitable for description of such models escape our understanding.

It is the second version that was used for all calculations, and the structures were all ***quarter-wave stacks***, i.e., thickness of individual layers satisfies the condition

$$4n_A d_A = 4n_B d_B = l_0$$

It was so chosen because the 'size' of an optical spectrum for a multilayer depends on the thickness of an individual layer and it is convenient when the spectra for different *N* are directly comparable, especially in the case of quarter-wave stacks when all computation can be restricted to one period $\omega\in[0, 2\omega_0]$, where $\omega_0=2\pi c/l_0$ However, from the physical point of view it makes no difference as all calculations were made in ***normalized*** frequency defined as $h\equiv\omega/\omega_0$.

Simple scaling relations allow to calculate the fractal dimensionality of any (*G*, **C**, *N*)-structure. It contains (*G*-*C*) previous generation substructures (*G*, **C**, *N-1*), each one taking up 1/G fraction in thickness. The structure being quarter-wave, the dimensionality equals

$$D=\frac{\ln(G-C)}{\ln G} \qquad (3).$$

The last thing to note in this section is that such a description of fractal multilayers structures, while being very general, is also a little superfluous. For instance, structures (3, {1}, *N*), (6, {2, 3}, *N*), (9, {3, 4, 5}, *N*) and so on are obviously identical in terms of normalized frequency. This can be written as a general rule that (*G*, **C**, *N*) and (*G*, **C′**, *N*) where **C**=$\{c_1,\ldots,c_m\}$ and $\mathbf{C}'=\left\{\{kc_1\ldots k(c_1+1)-1\}\cup\cdots\cup\{kc_m\ldots k(c_m+1)-1\}\right\}$ are identical from the spectral standpoint. Furthermore, structures (*G*, **C**, *kN*) and (G^k, **C′**, *N*) where **C′** can be derived from (*G*, **C**, *k*) are the same too (for instance, (3, {1}, 2) and (9, {1, 3, 4, 5, 7}, 1)), so the latter is also not unique. Additional identities appear if the refractive index of the medium surrounding the structure equals that of either material. Basically, **C** can be reduced to a set of numbers that would be more characteristic for fractal multilayers, but the technique of this reduction remains unclear. On the other hand, to the best of our knowledge, there are no known fractal multilayers that do not fit into the scheme presented.

3. Spectral scalability

Prior to discussion of the numerical results obtained, it is worth noting that all calculations were made using a method implemented by us [9] that is applicable for arbitrary multilayer stacks made up of isotropic, non-magnetic dielectric materials. The computation yields the optical transmission spectra as well as the field intensity distribution inside the structure for any given frequency.

As was already noted in Sec. 2, a (G, **C**, N)-structure contains several instances of previous generation substructures that are $1/G$ in thickness related to the whole stack. Taking into account self-similarity of fractal multilayers, this statement remains true for these (G, **C**, N-1) substructures which in turn encompass several (G, **C**, N-2)'s, and so on down to N=0. Thus, G^k can be called the *scaling factor* between the structures (G, **C**, N) and (G, **C**, N-k), and this property can be called *geometrical scalability* of fractal multilayers.

We have found this property has a *direct reflection* in the optical spectra. We have found that if one magnifies the central region (i.e., located around the central frequency) of the spectrum of a (G, **C**, N)-structure by a factor of G along the frequency axis, this central region will match the spectrum of a (G, **C**, N-1) almost perfectly. This *spectral scalability* was first noted for Cantor structures [9] (see Fig. 2 and 3) , but subsequent research reveals that other fractal multilayers also exhibit spectral scalability (see Fig. 4 and 5). What is more, the scaling factor is the same in this case and again equals G^k between (G, **C**, N) and (G, **C**, N-k).

The explanation is rather straightforward. Regarding a multilayer as a complex system of bound optical resonators (and there are all reasons for doing so, since fragments of fractal structures often have limited periodicity in a variety of scales, and periodic multilayers are known to be good mirrors), we can state that each resonator (e.g., a particular layer) introduces a mode (i.e., a resonant transmission peak) in the spectrum. The thicker such a layer is, the nearer the peak will be to the central frequency. Keeping this in mind, it is easy to see that it is *self-similarity* of fractal multilayers that results in spectral scalability, and not the geometrical scalability itself as it might seem at first sight.

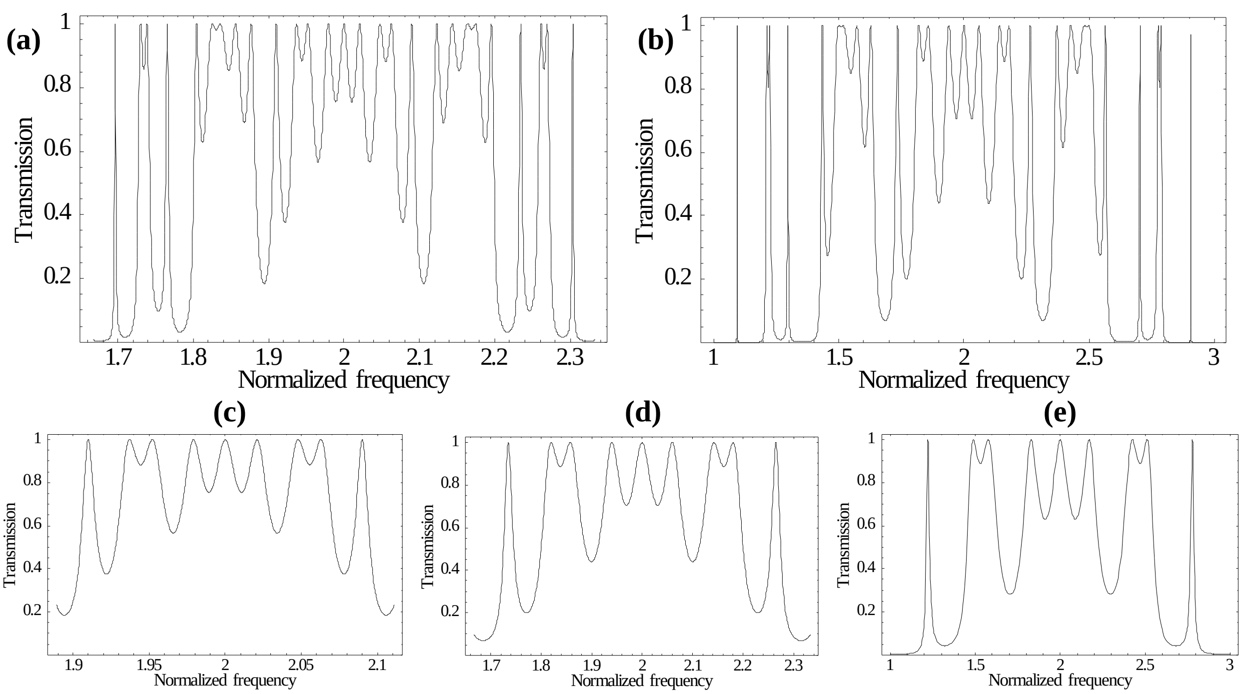


FIG. 2. Spectral scalability for Cantor (3,{1},*N*) structures: central part of the spectrum for *N*=4 scaled by three **(a)** versus full period of the spectrum for *N*=3 **(b)**; central part of the spectrum for *N*=4 scaled by 9 **(c)** and central part of the spectrum for *N*=3 scaled by three **(d)** versus full period of the spectrum for *N*=2 **(e)**. Comparing the graphs in the rows, one can notice apparent scaling properties.

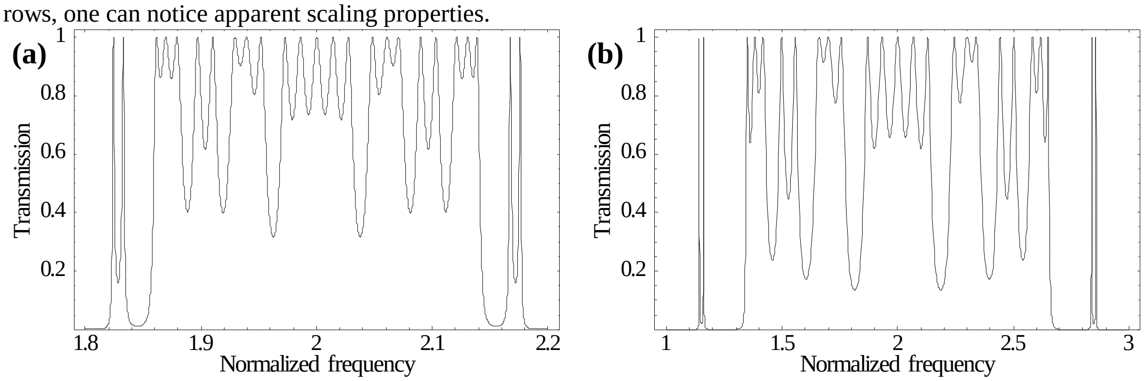


FIG. 3. Spectral scalability for Cantor (5,{1,3},*N*) structures: central part of the spectrum for *N*=3 scaled by 5 **(a)** versus full period of the spectrum for *N*=2 **(b)**. As in Fig. 2, scalability is seen clearly.

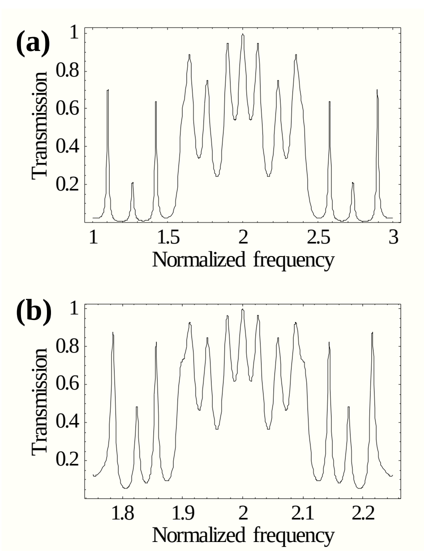


FIG. 4. Scaling properties of non-symmetric (4,{1},*N*) structures: the central part of the spectrum for *N*=4 scaled by 4 **(b)** versus full period of spectrum for (4,{1},3) **(a)**.

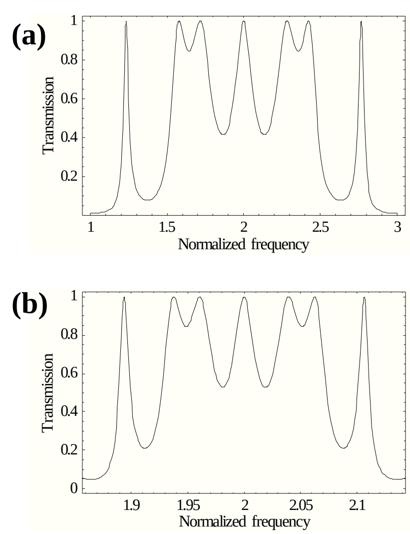


FIG. 5. Scaling properties of (7,{1,5},*N*) structures: the central part of the spectrum for *N*=2 scaled by 7 **(b)** versus full period of spectrum for *N*=1 **(a)**

4. Sequential splitting

However, geometrical scalability ***does*** result in another important property that can be treated as a direct correlation between fractal multilayers' structural and spectral features.

We have found that when the spectrum of a (*G*, **C**, *N*)-structure exhibits a resonant transmission peak on a certain frequency (and those peaks are very likely to be present in the spectra of most multilayers), in the next generation, i.e., in the (*G*, **C**, *N*+1)-spectrum, a multiplet replaces the peak, or, in other words, the peak ***splits*** into a multiplet. This splitting occurs every time *N* is increased, and we have therefore termed this property ***sequential splitting***. Between the multiplets that resulted from splitting new transmission peaks appear, also in multiplets in the general case. We have found out that the number of components in both kinds of the multiplets (call it ***split*** and ***new*** multiplets, respectively) is the same throughout the structure and solely depends on its generator.

Again, this property was first observed and is most apparent in Cantor structures (see Fig. 6), but other fractal multilayers also exhibit it to some extent (Fig. 7). One can see that the overall width of the multiplet remains almost unchanged during splitting, while the 'outer slopes' steepen, so the envelope becomes close to rectangular as *N* increases. This fact can have useful applications in optical band-pass and band-reject filters. On the contrary, new multiplets appear separated and do not have a common envelope.

Splitting can be explained using a resonant system approach, too. Previous generation substructures found in a (G, **C**, N) multilayer can be regarded as ***~~bound~~ coupled* *resonators***, and resonators are known to have their mode structure split when binding occurs. Additionally, according to the stack construction procedure, the increase of N introduces layers never found in previous generations, so new modes must appear in the spectrum as well, and that is what happens. From this reasoning it follows that the number of components in ***split*** multiplets m_{split} equals the number of previous generation substructures, while the number of components in ***new*** multiplets m_{new} equals the number of new layers introduced into the structure. For example, for Cantor structures (for which G is odd and **C**={1, 3,… G-1})

$$m_{\text{split}} = \frac{G+1}{2} \qquad (4),$$

$$m_{\text{new}} = \frac{G-1}{2} \qquad (5).$$

This can be additionally and directly confirmed by plotting the spatial distribution of the field intensity at the frequencies of the multiplet components in different cases. One can see in Fig. 9 that in the case of a split multiplet the field is localized in the previous generation substructures, while for new multiplets localization inside newly-introduced layers occurs. This shows what layers, or cavities from the bound resonators standpoint, are responsible for the peak (single or multiplet) at a given frequency. However, the *difference* between field profiles at the multiplet components is observed in other areas. For split multiplets such difference is found in newly-introduced layers, which act as binding areas. If new peaks appear in multiplets, too, then the difference between components' profiles is found between the layers where field is localized. This clearly demonstrates what regions of the structure are responsible for splitting and thus determine the number of components.

As regards other fractal multilayers, it can be seen that m_{split} is *always* the same for a given structure and equals

$$m_{\text{split}} = G - C \qquad (4).$$

The number of components in a new multiplet, however, may vary across the spectrum for certain structures. For instance, in a (5, {1, 2, 3}, N) new peaks appear as doublets *as well as* singles (see Fig. 7a). Additionally, in the spectra of many non-Cantor fractal multilayers new multiplets appear rather close to split ones, which makes them difficult to tell from each other, but the field intensity profiles would clear this confusion.

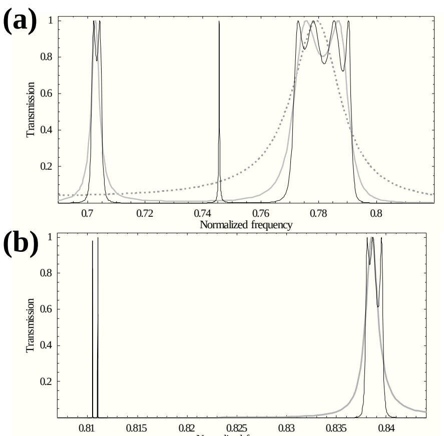


FIG. 6. Peak splitting in Cantor multilayers **(a)** for G=3 (*dashed*, N=2; *solid*, N=3; *thin solid*, N=4); **(b)** for G=5 (*thick*, N=2; *thin*, N=3). Note the number of components for split and new multiplets and the unchanged width of the multiplet during splitting.

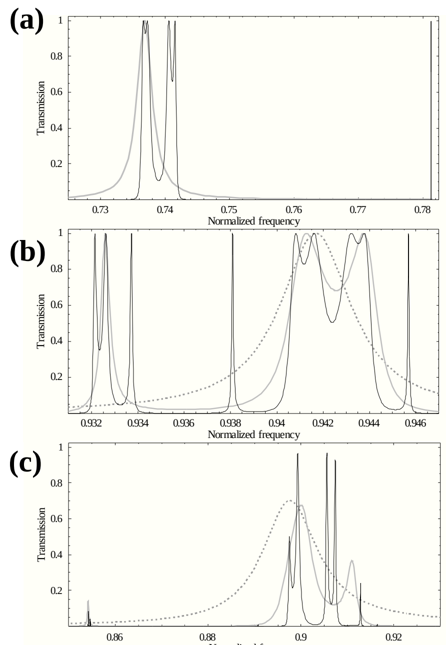


FIG. 7. Peak splitting in non-Cantor fractal multilayers: **(a)** (5,{2},N), N=2 and 3; **(b)** (5,{1,2,3},N), N=2,3,4; **(c)** (4,{1},N), N=2,3,4. The line dashing and thickness conventions are the same as in Fig. 6. Note a very narrow ($\Delta\eta \cong 1.5 \cdot 10^{-7}$) single peak at the right edge of **(a)**.

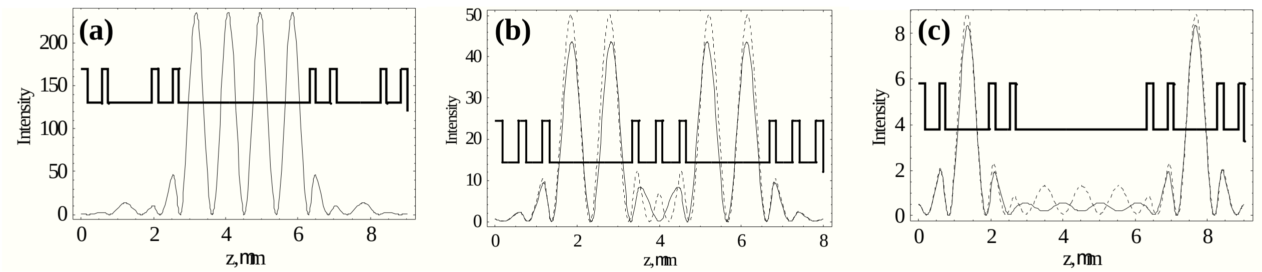


FIG. 8. Field intensity profiles at the frequencies of the peaks: **(a)** (3,{1},3), new single peak; **(b)** (5,{1},3), new doublet (*solid* and *dashed* correspond to two components); **(c)** (3,{1},3), split doublet. Localization regions and areas where the components' profiles differ can be observed.

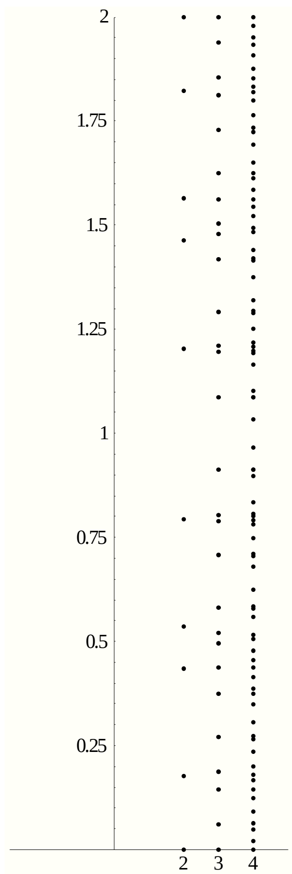


FIG. 9. A resonance map (all transmission peaks, normalized frequency versus generation number) for (3,{1},N) structures.

5. Conclusions

As a main result of this work, we can state that a direct correlation between geometrical and spectral properties of multilayer fractal structures exists and manifests itself in at least two distinctive effects, *spectral scalability* and *sequential peak splitting.* It is then natural to put forward another question on whether geometrical *fractality* of the structures under study is present in their optical spectra owing to the correlation found.

Fractal spectral properties, if any, are most likely to be found in the *transmission band spectrum* (also called *resonance map*), i.e., the locations of resonance peaks (see Fig. 10 for an example with Cantor structures). On the one hand, there *are* known multilayers with fractal transmission band spectra [5, see also references therein], and it can be seen from Fig. 9 that as N increases, new peaks appear according to a definite pattern, which points towards the presence of fractal dimensionality in the spectra. On the other hand, one can easily notice that new resonances appear between *all* the existing ones when N increases, the whole period being filled with peaks to a greater or a less extent. Therefore it is obvious that the fractal such spectra might represent have nothing in common with a Cantor fractal set, which is the case with quasi-periodic multilayers and which would be most logical for Cantor multilayers from the point of view of the above mentioned correlation. Besides, there are indications [12] that the peak locations for Cantor structures at least can be determined with a series of periodic functions with the minimal period inversely proportional to N, so the peaks are expected to fill the frequency scale uniformly when N approaches infinity. Nevertheless, in this latter case not only will resonance peaks appear everywhere in the spectrum, but also regions of zero transmission be found between most of them. So, the question on fractal behavior of the spectra of multilayer fractal structures remains a topic for future investigations.

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